GROUP-THEORETICAL DETERMINATION OF THE MIXING ANGLE IN THE ELECTROWEAK GAUGE GROUP*

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Abstract. The assumption that the Weinberg rotation between the gauge fields associated with the third component of the "weak isospin" (T_3) and the hypercharge (Y) proceeds in a natural way from a global homomorphism of the $SU(2)\otimes U(1)$ gauge group in some locally isomorphic group (which proves to be $U(2)$), imposes strong restrictions so as to fix the single value $\sin^2\theta_W = 1/2$. This result can be thought of only as being an asymptotic limit corresponding to an earlier stage of the Universe. It also lends support to the idea that e^2/g^2 and $1 - M_W^2/M_Z^2$ are in principle unrelated quantities.

There are two basic ingredients in the constitution of a model to describe the unified electroweak interactions, the Weinberg-Salam-Glashow Standard Model, which deserve

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further study and which lessen the (mathematical) beauty of the theory as a whole. One is the way in which the W-Z-bosons acquire mass, the Higgs mechanism, and the other is the rotation between the gauge fields associated with the third component of weak isospin (A^3_μ) and the hypercharge (A^4_μ), intended to define the proper electromagnetic field, without any (apparent) connection to the “weak” Gell'man-Nishijima relation

$$Q = T_3 + \frac{1}{2}Y \,, \tag{1}$$

meant to define a proper electric charge in the Lie algebra. We shall focus on the latter question.

In this paper we wish to explore the restrictions that appear on the mixing angle θ_W as a consequence of the natural consistence requirement that the rotation in the gauge fields

$$\begin{aligned} Z^0_\mu &= \cos\theta_W A^3_\mu - \sin\theta_W A^4_\mu \\ A_\mu &= \sin\theta_W A^3_\mu + \cos\theta_W A^4_\mu \end{aligned} \tag{2}$$

comes from an exponentiable (linear) transformation on the Lie algebra of $SU(2)\otimes U(1)$. Since the gauge group is not simply connected, it is not true that any automorphism of the Lie algebra can be realized as the derivative of a global group homomorphism or, in other words, a differentiable mapping between two locally isomorphic groups providing a given automorphism of the (commom) Lie algebra, can in general destroy the global group law.

To analyse the set of global homomorphism from $SU(2)\otimes U(1)$ to a locally isomorphic group we can proceed in two different ways: either we study the set of discrete normal subgroups of $SU(2)\otimes U(1)$, which are the possible kernels of those homomorphisms, or we write the explicit group law of $SU(2)\otimes U(1)$, perform an arbitrary homomorphism

and analyze the conditions under which the group law is not destroyed. We shall follow the second approach although some comments on the first one will be added at the end.

Let us parametrize the group $SU(2)$ in a co-ordinate system adapted to the Hopf fibration $SU(2) \to S^2$, the sphere S^2 being parametrized by stereographic projection. The $SU(2) \otimes U(1)$ group law in the local chart at the identity, which nevertheless keeps the global character of the toral subgroup, is:

$$
\begin{aligned}
\eta'' &= \frac{z_1''}{|z_1''|} = \frac{\eta'\eta - \eta'\eta^* C' C^*}{\sqrt{(1-\eta^{*2}C'C^*)(1-\eta^2 C C^{*\prime})}} \\
C'' &= \frac{z_2''}{z_1''} = \frac{C\eta^2 + C'}{\eta^2 - C'C^*} \\
C^{*\prime\prime} &= \frac{z_2''^*}{z_1''^*} = \frac{C^*\eta^{-2} + C^{*\prime}}{\eta^{-2} - C^{*\prime}C} \\
\zeta'' &= \zeta'\zeta
\end{aligned}
\tag{3}
$$

where $\eta \in U(1) \subset SU(2), \zeta \in U(1), C \in \mathbf{C}$ and z_1, z_2 characterize a $SU(2)$ matrix $\begin{pmatrix} z_1 & z_2 \\ -z_2{}^* & z_1{}^* \end{pmatrix}$. The commutation relations between the (right) generators $T_+ \equiv X_{C^*}$, $T_- \equiv X_C, T_3 \equiv X_\eta$ and $Y \equiv X_\zeta$ are:

$$
\begin{aligned}
[T_3, T_\pm] &= \pm 2T_\pm \\
[T_+, T_-] &= T_3 \\
[Y, \mathrm{all}] &= 0
\end{aligned}
\tag{4}
$$

We shall consider transformations induced by an homomorphism of the torus:

$$
\tilde{\eta} = \eta^p \zeta^{p'}
$$

$$
\begin{aligned}
\tilde{\zeta} &= \eta^{q}\zeta^{q'} \\
\tilde{C} &= C\,,\ \tilde{C^*} = C^*,
\end{aligned} \tag{5}
$$

where the parameters p, p', q, q' have to be integers for the univalueness requirement.

After we apply this transformation the group law becomes:

$$
\begin{aligned}
\tilde{\eta}'' &= \left(\frac{\tilde{\eta}'^{\frac{1}{p}}\tilde{\eta}^{\frac{1}{p}} - \tilde{\eta}'^{\frac{1}{p}}\tilde{\eta}^{-\frac{d+2qp'}{dp}}\tilde{\zeta}^{\frac{2p'}{d}}\tilde{C^*}\tilde{C}'}{\sqrt{(1-\tilde{\eta}^{-\frac{2q'}{d}}\tilde{\zeta}^{\frac{2p'}{d}}\tilde{C}'\tilde{C^*})(1-\tilde{\eta}^{\frac{2q'}{d}}\tilde{\zeta}^{-\frac{2p'}{d}}\tilde{C}\tilde{C^*}')}} \right)^{p} \\
\tilde{C}'' &= \frac{\tilde{C}\tilde{\eta}^{\frac{2q'}{d}}\tilde{\zeta}^{-\frac{2p'}{d}} + \tilde{C}'}{\tilde{\eta}^{\frac{2q'}{d}}\tilde{\zeta}^{-\frac{2p'}{d}} - \tilde{C}'\tilde{C^*}} \\
\tilde{C^*}'' &= \frac{\tilde{C^*}\tilde{\eta}^{-\frac{2q'}{d}}\tilde{\zeta}^{\frac{2p'}{d}} + \tilde{C^*}'}{\tilde{\eta}^{-\frac{2q'}{d}}\tilde{\zeta}^{\frac{2p'}{d}} - \tilde{C^*}'\tilde{C}} \\
\tilde{\zeta}'' &= \tilde{\zeta}'\tilde{\zeta}(\tilde{\eta}''\tilde{\eta}'^{-1}\tilde{\eta}^{-1})^{\frac{q}{p}}\ ,
\end{aligned} \tag{6}
$$

where d is the determinant of the matrix $\begin{pmatrix} p & p' \\ q & q' \end{pmatrix}$, and this group law is well-behaved if

$$
\frac{2p'}{d} = m,\ \frac{2q'}{d} = n,\ \frac{q}{p} = k,\ m, n, k \in Z \tag{7}
$$

which, in particular, imply $p = \pm 1, \pm 2$. This particular result simply states the well-known fact that the only invariant subgroups of $SU(2)$ itself are I (the identity) and Z_2, respectively.

The commutation relations between the new generators (with a definition analogous to that given above),

$$
\begin{aligned}
\left[\tilde{T}_3, \tilde{T}_{\pm}\right] &= \pm\frac{2q'}{d}\tilde{T}_{\pm} \\
\left[\tilde{Y}, \tilde{T}_{\pm}\right] &= \pm\frac{-2p'}{d}\tilde{T}_{\pm} \\
\left[\tilde{T}_{+}, \tilde{T}_{-}\right] &= p\tilde{T}_3 + q\tilde{Y}\ ,
\end{aligned} \tag{8}
$$

can be obtained directly from (7) or by applying the tangent mapping to (5) to the old ones. This transformation gives:

$$\begin{aligned}\tilde{T}_3 &= \frac{q'}{d}T_3 - \frac{q}{d}Y \\ \tilde{Y} &= \frac{-p'}{d}T_3 + \frac{p}{d}Y\end{aligned} \tag{9}$$

and provides a generalized Gell'Mann-Nishijima relation and its counterpart, which now appear quantized.

Let us now examine the transformation induced by (5) in the ($3^{rd} - 4^{th}$ internal components of the) gauge fields. It is given by:

$$\begin{pmatrix} \tilde{A}^3_\mu \\ \tilde{A}^4_\mu \end{pmatrix} = \begin{pmatrix} \frac{1}{\tilde{r}} & 0 \\ 0 & \frac{1}{\tilde{r}'} \end{pmatrix} \begin{pmatrix} p & p' \\ q & q' \end{pmatrix} \begin{pmatrix} r & 0 \\ 0 & r' \end{pmatrix} \begin{pmatrix} A^3_\mu \\ A^4_\mu \end{pmatrix} \tag{10}$$

where r, r' are the original coupling constants associated with isospin and hypercharge respectively, and $\tilde{r}, \tilde{r}'$ are the final ones. In fact, the covariant derivative $D_\mu = \partial_\mu - ig^k_i T_k A^i_\mu$, where i, k run over 1,2,3,4 ($T_4 \equiv Y$) goes to $\tilde{D}_\mu = \partial_\mu - i\tilde{g}^k_i \tilde{T}_k \tilde{A}^i_\mu = D_\mu$. Therefore,

$$\tilde{A}^l_\mu = (\tilde{g}^{-1})^l_j \, a^j_k \, g^k_i \, A^i_\mu \tag{11}$$

where a^j_k is the transformation matrix changing co-ordinates in the Lie algebra, which contains the central matrix in (10) as a box, and $\mathbf{g} = \mathrm{diag}(r, r, r, r')$ and $\tilde{\mathbf{g}} = \mathrm{diag}(\tilde{r}, \tilde{r}, \tilde{r}, \tilde{r}')$ are the initial and final (bare) coupling constants matrices.

We now impose the requirement that the complete transformation (10), rather than the central matrix in it, be the Weinberg rotation (3) ($Z^0_\mu \equiv \tilde{A}^3_\mu, A_\mu \equiv \tilde{A}^4_\mu$). This results in

$$\frac{\tilde{r}^2}{\tilde{r}'^2} = -\frac{pp'}{qq'}\,, \;\; \frac{r^2}{r'^2} = -\frac{q'p'}{qp}\,, \;\; \tan^2\theta_W = -\frac{qp'}{pq'}\,, \;\; \tilde{r} = \frac{p}{\cos\theta_W}\, r\,, \tag{12}$$

which contain a further restriction: the product of the four integers $pp'qq' < 0$, a condition afterwards necessary to have a (non-trivial) rotation. If the transformation (5) is an automorphism of the torus ($d = \pm 1$), then the only possible rotations between the gauge fields are the trivial ones ($\tan^2\theta_W = 0, \infty$), so that the final group has to be the quotient of $SU(2)\otimes U(1)$ by a non-trivial normal (discrete) subgroup. Adding (12) to (7) we arrive at the final result:

$$\{p = \pm 1 \text{ and } (p' = -kq', k = \pm 1)\} \Rightarrow \left\{\tan^2\theta_W = 1\,,\, d = \pm 2q'\right\} \tag{13}$$

For these values of p, p', q, q' the kernel of the homomorphism (see the transformation (5)) is the normal subgroup

$$H_d \equiv \left\{(C, C^*, \eta; \zeta) = (0, 0, 1; e^{i\frac{2s}{d}2\pi}), (0, 0, -1; e^{i\frac{2s+1}{d}2\pi})/s = 0, 1, ..., \frac{|d|}{2} - 1\right\} \tag{14}$$

which is isomorphic, as a group, to $Z_{|d|}$. All these homomorphisms lead to the same value for $\tan^2\theta_W (= 1)$ and indeed, all can be written as:

$$\begin{pmatrix} p & p' \\ q & q' \end{pmatrix} = \begin{pmatrix} \pm 1 & -kq' \\ \pm k & q' \end{pmatrix} = \begin{pmatrix} 1 & -k \\ k & 1 \end{pmatrix}\begin{pmatrix} \pm 1 & 0 \\ 0 & q' \end{pmatrix} \tag{15}$$

where the second factor has determinant $\pm q'$, and represents a transformation from $SU(2)\otimes U(1)$ to $SU(2)\otimes(U(1)/Z_{|q'|})$, and the first one has determinant 2 and would take $SU(2)\otimes U(1)$ to $(SU(2)\otimes U(1))/H_2 \approx U(2)$ by itself. The second factor affects the quotient between the original coupling constants (not the final one), as can be seen in (12), and the generalized Gell'Man-Nishijima relation (9). Among the possible values for q' only $q' = \pm 1$ provides us with a proper electric charge; the choice of the signs of p, q, p' is a matter of convention and will define either $\tilde{T}_3$ or $\tilde{Y}$ as $\pm$ the electric charge Q. The corresponding homomorphism has Kernel $H_2 = \{(0, 0, 1; 1), (0, 0, -1; -1)\}$ and $U(2)$ as the image group (true gauge group) [2, 3, 4].

With the usual choice of multiplets in the Lagrangian of the Standard Model (see e.g. [5]) T_3 and Y have the expressions

$$T_3 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \; Y = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 2 \end{pmatrix} \tag{16}$$

which agrees with the usual expressions if the $U(1)$ subgroups are trivially reparametrized by $\alpha = -2i\ln\eta, \beta = i\ln\zeta$ $(T_3 \to \frac{1}{2}T_3, Y \to -Y)$. The particular choice of signs $p = p' = q' = -q = -1$ yields:

$$Q = \tilde{Y} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{pmatrix}, \; \tilde{T_3} = \begin{pmatrix} -1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -1 \end{pmatrix} \tag{17}$$

The first surprising result is the fact that only one value of $\tan^2\theta_W$ is allowed, which means only one coupling constant (the electric charge, essentially, i.e. $e \equiv \tilde{r}' = \sqrt{2}r \equiv g/\sqrt{2}$), even though the gauge group ($U(2)$) is not a simple group. According to general settings [6], however, the theory must contain a coupling constant for each simple or abelian term in the Lie algebra decomposition. An immediate conclusion is that the assignment of constants should be done according to factors in the direct product decomposition of the group, rather than the algebra.

The second result is the particular structure of the neutral weak current derived from the expression of $\tilde{T_3}$ above, according to which the gauge field Z^0 interacts with the (left-handed) neutrino and the right-handed electron only; i.e. the neutral weak current is pure V-A for the neutrino and pure V+A for the electron.

Last, but not least, is the striking value of $\sin^2\theta_W = \frac{1}{2}$ $(\tan^2\theta_W = 1)$, far from the experimental value ≈ 0.23 [7]. In the light of this result, only the hope remains that our theoretical value of θ_W really corresponds to that state of the Universe in which the

electroweak interaction was not yet spontaneously broken, and that the process of spontaneous symmetry breaking, not fully understood (at least from a pure group-theoretical point of view) could relax the strong conditions (7). For instance, breaking down the $SU(2)$ group law (7) and preserving the $U_{em}(1)$, leads to $\tan^2\theta_W = -\frac{q}{q'}$, allowing any rational value.

In any case, the discrepancy between our theoretical value for $sin^2\theta_W$ as given by a ratio of coupling constants, $\frac{e^2}{g^2} = \frac{\tilde{r}'^2}{(2r)^2}$, and the experimental one obtained through the expression $1 - \frac{M_W^2}{M_Z^2}$, lends support to the idea that in principle both quantities are not related [8].

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